DISCHARGE OF GAS INTO VACUUM WITH TEMPERATURE AT THE BOUNDARY SUBJECT TO EXPONENTIAL LAW

(ISTECHENIE GAZA V VAKUUM PRI STEPENNOM ZAKONE TEMPERATURY NA GRANITSE)

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A two-dimensional self-similar problem of discharge of a heat conducting gas into vacuum is analyzed. The temperature at the boundary of gas and vacuum is assumed to change as an exponential function of time. The coefficient of thermal conductivity depends exponentially on temperature and density. The initial gas density is assumed to be finite and constant. With definite values of exponents this problem is self-similar, i.e. the system of partial differential equations can be reduced to the solution of a system of ordinary equations.

The self-modeling properties of solutions of this kind of problems has been noted earlier in [1 and 2]. The problem analyzed here is a particular case of the problem of piston motion considered in [3]. In this problem, however, there appears at the boundary of gas and vacuum a new singular point which does not occur in the piston problem.

A numerical solution of the boundary value problem defined by a system of ordinary equations is made difficult by the presence in the latter of singular points, and of discontinuities in the sought solution. These difficulties have been overcome by a qualitative analysis of the behavior of integral curves, and by the selection of a suitable method of numerical integration.

It is shown in this work that, depending on the initial parameters of the problem, there may exist two kinds of solutions. This had been noted earlier in [1, 3 and 4]. Examples of these are presented here. The degeneration of the solution into a trivial one, when the thermal conductivity coefficient is either invariant of density, or increases with increasing density, is pointed out.

1. We consider a cold gas defined by the following equations of state:

$$p = R\rho T, \qquad \varepsilon = \frac{R}{\gamma - 1} T$$

Here, R is the gas constant, γ is the Poisson's adiabatic exponent, p is the pressure, ρ the density, T the temperature, and ε the internal energy. One-dimensional equations of a plane motion in Euler coordinates have the form

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)+\frac{\partial p}{\partial r}=0, \qquad \frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial r}=0$$

$$\frac{\partial}{\partial t}\left(\varepsilon+\frac{u^2}{2}\right)+u\frac{\partial}{\partial r}\left(\varepsilon+\frac{u^2}{2}\right)+\frac{1}{\rho}\frac{\partial}{\partial r}\left(pu-\varkappa\frac{\partial T}{\partial r}\right)=0 \qquad (1.1)$$

Here u is the velocity and $\varkappa$ the coefficient of thermal conductivity. The temperature of the vacuum space bounding the gas is given by

$$T=T_0t^n \qquad (n>0,\quad T_0=\mathrm{const}) \qquad (1.2)$$

With the assumption that

$$\varkappa=\varkappa_0\rho^{-l}T^{(1+n)/n} \qquad (\varkappa_0=\mathrm{const})$$

the problem will be a self-similar one. The introduction of variables

$$\lambda=\frac{(n+2)r}{2\sqrt{RB_0}t^{1+{}^1/_2n}}, \qquad T=B_0t^n\theta(\lambda) \qquad (B_0=\mathrm{const}\,[T^\circ])$$

$$u=\sqrt{RB_0}t^{{}^1/_2n}\zeta(\lambda), \qquad \rho=\rho_0\delta(\lambda) \qquad (1.3)$$

reduces system (1.1) to a system of ordinary equations

$$\theta'=-\frac{\delta^l}{A}\theta^{-\frac{1+n}{n}}q, \quad \delta'=\frac{\delta\zeta'}{\lambda-\zeta}, \quad A=\frac{\varkappa_0}{R^2\rho^{l+1}}B_0^{\frac{1}{n}}$$

$$\zeta'=\left(\theta'+\frac{n}{n+2}\zeta\right)\frac{\lambda-\zeta}{(\lambda-\zeta)^2-\theta} \qquad (1.4)$$

$$q'=-\frac{2\delta}{(2+n)(\gamma-1)}\left[(\gamma-1)\theta\zeta'+\frac{2n}{n+2}\theta+(\zeta-\lambda)\theta'\right]$$

where A is a dimensionless constant.

As the front of expansion is also the trajectory of particles, we have

$$u=\frac{dr}{dt}=\sqrt{RB_0}t^{{}^1/_2n}\lambda_0$$

Taking into account (1.2) and (1.3) we obtain

$$\theta_0=T_0/B_0, \qquad \delta_0=0, \qquad \lambda_0=\zeta_0 \qquad (1.5)$$

It was shown in [5 and 6] in which gas dynamic effects were omitted that with $n>0$ the perturbation front spreads with a finite velocity. Here, it is assumed that $\lambda<\infty$. The value of B_0 can always be selected so as to have th the perturbation front $\lambda=1$, in which case we have

$$\text{(a)}\ \theta=0, \quad \text{(b)}\ \zeta=0, \quad \text{(c)}\ \delta=1, \quad \text{(d)}\ q=0 \qquad (1.6)$$

Conditions (1.6) mean that at the perturbation front the hydrodynamic parameters and the heat flux are continuous. If $\theta'(1)<\infty$, then condition (d) follows from (a). However, generally speaking, the condition for $\theta'(1)<\infty$ is not always fulfilled.

2. In deriving our solution, we shall use the conditions at the discontinuity which in the presence of thermal conductivity effects are

$$\rho_1(u_1 - D) = \rho_2(u_2 - D),\ \ \rho_1(u_1 - D)^2 + R\rho_1 T_1 = \\ = \rho_2(u_2 - D)^2 + R\rho_2 T_1$$

$$\rho_1(u_1 - D)\left(\frac{u_1^2}{2} + \frac{R}{\gamma - 1}T_1\right) + R\rho_1 T_1 u_1 - \varkappa_1 \frac{\partial T}{\partial r}\Big|_1 = \\ = \rho_2(u_2 - D)\left(\frac{u_2^2}{2} + \frac{R}{\gamma - 1}T_1\right) + R\rho_2 T_1 u_2 - \varkappa_2 \frac{\partial T}{\partial r}\Big|_2 \tag{2.1}$$

Here, D is the discontinuity velocity, and the subscripts 1 and 2 denote parameters in front of the discontinuity, and behind it respectively. In the derivation of equations we shall make use of the fact that the temperature in the interval between the forward and the rear discontinuity fronts is continuous (isothermic process).

As the problem considered here ia a self-similar one, the discontinuity will always be a λ-line. Therefore,

$$D = \sqrt{RB_0}\, t^{1/2n} \lambda_1 \tag{2.2}$$

The substitution of (1.3) and (2.2) into (2.1) yields

$$\zeta_2 = \lambda_1 + \frac{\theta_1}{(\zeta_1 - \lambda_1)}, \qquad \delta_2 = \frac{\delta_1(\zeta_1 - \lambda_1)^2}{\theta_1}, \qquad q_2 = q_1 + \frac{\gamma\delta_1}{2+n}\frac{(\zeta_1 - \lambda_1)^4 - \theta_1^2}{\zeta_1 - \lambda_1} \tag{2.3}$$

3. Point $M\ (\lambda = 1,\ \theta = 0,\ \zeta = 0,\ \delta = 1,\ \dot{q} = 0)$, which corresonds to the perturbation front (1.6) will be a singular point. It belongs to a manifold of singular points $\theta = 0$, $q = 0$. The integral curves passing through point M can be found by separating the main terms of the right-hand sides of system (1.4). The singular point M is such that only one integral curve emanating from it can be found in the relevant area. The solution in the perturbation front neighborhood is approximately defined by the following Formulas:

$$\dot{k} = \left[\frac{2(n+1)}{A(n+2)(\gamma-1)n}\right]^{\frac{n}{n+1}}, \qquad \theta = k(1-\lambda)^{\frac{n}{n+1}}, \qquad \delta = e^{\xi}$$

$$\zeta = \frac{nk}{n+1}\exp\frac{n\lambda}{n+2}\int_1^{\lambda}(1-\lambda)^{\frac{-1}{n+1}}\exp\frac{-n\lambda}{n+2}\,d\lambda \tag{3.1}$$

We note that with $n > 0$ we have $n/(n+1) < 1$ which means that at the front the temperature derivative is infinite.

4. It is known that in the absence of thermal conductivity, the gas efflux velocity is finite and equal to $-2c_0/(\gamma - 1)$, where c_0 is the initial velocity of sound of the gas at rest. It was noted in [7] that the efflux velocity of a gas subject to energy release, is infinite. In the problem considered here, the heat conducting gas will also expand with infinite velocity. We shall prove this (*).

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*) The proof of this in Lagrangean coordinates was kindly communicated to the author by S.P. Kurdiumov and P.P. Volosevich.

We shall show that $\delta(\lambda_0) \neq 0$, if $\zeta_0 = \lambda_0$ is of finite magnitude. It follows from Equations (1.4) that

$$(\ln \delta)' = \left(\theta' + \frac{n}{n+2}\zeta\right)[(\lambda-\zeta)^2-\theta]^{-1} \equiv f(\lambda)$$

Integrating the above equation over certain interval $\lambda_0 \le \lambda \le \lambda_1$. we obtain

$$\delta(\lambda) = \delta(\lambda_1)\exp - \int_{\lambda}^{\lambda_1} f(\lambda)\,d\lambda \tag{4.1}$$

We divide the integral of (4.1) into two

$$J(\lambda) = \int_{\lambda}^{\lambda_1} f(\lambda)\,d\lambda = -\frac{n}{n+2}\int_{\lambda}^{\lambda_1}\frac{\zeta\,d\lambda}{(\lambda-\zeta)^2-\theta} - \int_{\lambda}^{\lambda_1}\frac{\theta'\,d\lambda}{(\lambda-\zeta)^2-\theta}$$

and consider $\lim J$ for $\lambda \to \lambda_0 + 0$. It is evident that the first integral is finite, and so is the second one. It can, in fact, be replaced in the neighborhood of point λ_0 by

$$-\int_{\lambda_0}^{\lambda_1}\frac{\theta'\,d\lambda}{(\lambda-\zeta)^2-\theta} = c_2 + \int_{\lambda_0}^{\lambda_1}\frac{\theta'}{\theta}\,d\lambda = c_2 + \ln\frac{\theta(\lambda_1)}{\theta(\lambda_0)}$$

Here c_2 is a finite constant. Therefore, the integral

$$J(\lambda_0) = \lim_{\lambda=\lambda_0+0} J(\lambda) = c_1 + c_2 + \ln\frac{\theta(\lambda_1)}{\theta(\lambda_0)}$$

is finite and $\delta(\lambda_0) \neq 0$, which contradicts condition (1.5). Therefore, the expansion velocity cannot be finite. We may point out that this proof is valid for any λ .

It is possible to indicate an integral curve for which $\delta_0 = 0$, if we assume $\zeta_0 = -\infty$. Proof of this will be given in Section 6.

5. The expression for a dimensionless flow is given by the formula of (1.4)

$$q = -A\delta^{-l}\theta^{\frac{1+n}{n}}\theta'$$

To have a physical sense, the flow at the front of expansion must be finite and positive. Therefore,

$$\theta'(-\infty) = 0 \quad (l > 0); \qquad \theta'(-\infty) < 0 \neq \infty \quad (l = 0)$$

$$\theta'(-\infty) = -\infty \ (l < 0) \tag{5.1}$$

Because of the condition that $\theta'(-\infty) \neq 0$, when $\lambda \le 0$ and $\theta(\lambda) \ge 0$ for $-\infty < \lambda \le 1$, it follows that $\theta_0 = \infty$. Consequently, the problem stated in Section 1 can have any meaning only, if the temperature at the boundary is considered to be infinitely great. This means that with the exponential temperature law (1.2), and with a finite value of T_0 and $\lambda \le 0$, the problem has a degenerate solution. The boundary expanding with an infinite velocity, locks the heat flow at infinity, and there is no motion whatsoever.

6. Conditions at the front of expansion (1.5) are defined by point

$$N\,(\lambda = -\infty,\ \theta = \theta_0,\ \zeta = -\infty,\ \delta = 0,\ q = q_0) \tag{6.1}$$

where θ_0 and q_0 remain to be determined. Point N, as point M, is a singular point of system (1.4), and belongs to a manifold of singular points

$$\lambda = -\infty, \qquad \zeta = -\infty, \qquad \delta = 0$$

It follows from conditions (6.1) and (5.1) that $\theta_0' = 0$, when $\lambda > 0$. It is, therefore, possible to substitute, as an approximation of ζ' of (1.4) in the neighborhood of point N, by Equation

$$\zeta' = \frac{n\zeta(\lambda - \zeta)}{(n+2)\,[(\lambda - \zeta)^2 - \theta_0]} \tag{6.2}$$

We shall use the transformation $\zeta = 1/y$, $\lambda = 1/x$ for the analysis of integral curves in the neighborhood of point $(x = 0,\ y = 0)$ within quadrant $x \leq 0$, $y \leq 0$. Equation (6.2) then becomes

$$y' = \frac{n\,(y - x)\,y^2}{(n+2)\,[(x - y)^2 - \theta_0 x^2 y^2]} \tag{6.3}$$

and the singular point moves into the coordinate origin. Its character is established on the basis of results obtained by Frommer (8). We introduce notations

$$y\,/\,x = u, \qquad \psi\,(u,\,x) = y'\,(x,\,u) - u$$

The critical directions along which the integral curves may, in this case, reach the singular point will be

$$(1)\ u = 0, \qquad (2)\ u = 1, \qquad (3)\ u = (n+2)\,/\,n, \qquad (4)\ u = \infty$$

According to [8] the following qualitative conclusions may be deduced. There is only one intgral curve in direction (1), namely, the coordinate axis $y = 0$ (because $\partial\psi/\partial u < 0$, when $x = 0,\ u = 0$). There are also single curves only with slopes defined by (2) and (3) which pass through the coordinate origin ($\partial\psi/\partial u < 0$, with $u = 1$, $x = 0$, and $\partial\psi/\partial u < 0$, with $u = (n+2)/n$, $x = 0$). Direction (4) has an infinite multiplicity of integral curves. The character of curves defined by Equation (6.3) is represented in Fig.1.

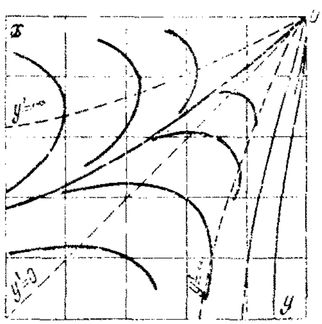

Fig. 1

The curve with slope (2) will yield the unknown solution, and its equation may be expanded into the series

$$y = x - \theta_0\,\frac{n+2}{n}\,x^3 + \ldots \tag{6.4}$$

Solutions (3) and (4) must be discarded, as their substitution into (1.4) yields negative values of density in the neighborhood of point N.

It can be shown with the use of expansion (6.4) and of the expression for δ' of (1.4) that $\lim \delta(\lambda) = 0$, when $\lambda = -\infty$, which means that boundary condition (1.5) is fulfilled.

7. The solution of the system of Equations (1.4) is dependent on the dimensionless parameter A. Its variation depends on parameter $\varkappa_0$. With $A = \infty$, it can be assumed that

$$\theta'(\lambda) = 0 \tag{7.1}$$

and consider Equation (6.2). If in (1.5) $B_0 = T_0$ is assumed, then $\theta(\lambda) = 1$ for all $\lambda \in (-\infty, \infty)$. Equation (6.2) is to be solved with the following boundary conditions: ($\lambda = -\infty$, $\zeta = -\infty$) and ($\lambda = \infty$, $\zeta = 0$). Point ($\lambda = \infty$, $\zeta = 0$) in the case of Equation (6.2) is an isolated singular point of the "saddle" type. Axis $\zeta = 0$ will be the solution when $1 \le \lambda < \infty$. Point ($\lambda = 1$, $\zeta = 0$), which is of nodal type, corresponds to a weak discontinuity. The solution in the interval $(-\infty, 1)$ is determined by numerical methods.

Thus, when $A = \infty$, a heat wave heats momentarily the whole volume of gas, and the boundary between gas and vacuum recedes into infinity, while a relaxation wave travelling with the isothermal velocity of sound $\sqrt{RT_0}t^{1/2n}$ spreads throughout the quiescent gas.

8. With $A \neq \infty$, the problem is reduced to finding a solution which would satisfy Equations (1.4) and boundary conditions (3.1) and (6.1). The problem is solved by numerical methods. Integration from $\lambda = 1$ to $\lambda = -\infty$ yields, as in the case of the problem considered in [4], a point λ°, ζ° at which $d\lambda/d\zeta = 0$, $d^2\lambda/d\zeta^2$. The singularity arising in this case prevents the derivation of a continuous solution. A discontinuous solution is, consequently, formulated, using Equations (2.3) at discontinuity. The independent parameter λ obtained in the process makes it possible to satisfy one of the conditions (6.1), namely: $\zeta = -\infty$, when $\lambda = -\infty$. The condition of $\delta_0 = 0$ is automatically fulfilled.

9. The above assumptions tend to indicate the existence of a certain value of $A = A_*$ which would segregate all solutions into two classes.

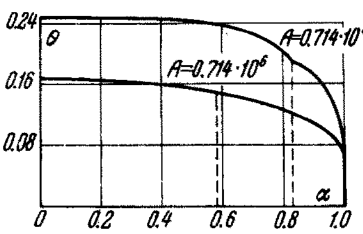

Fig. 2

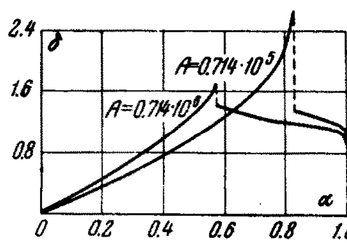

Fig. 3

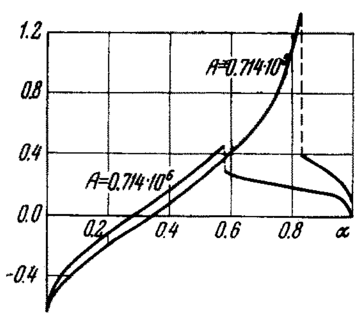

Fig. 4

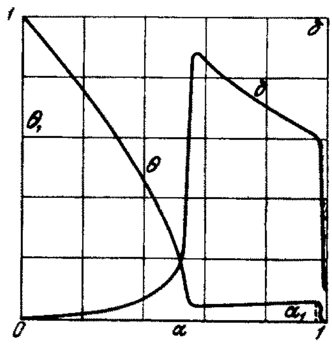

Fig. 5

With $A_* < A < \infty$ the temperature $\theta(\lambda)$ will be monotonous in the interval $(-\infty, \lambda_1]$, with $\min\theta$ at point λ_1. The qualitative features of this solution may be described as follows: throughout a gas at rest with $t > 0$ there spreads a perturbation at the front of which the parameters T, ρ, u and $\varkappa(\partial T/\partial r)$ undergo a continuous change, with only the derivatives of these becoming discontinuous. A perturbation of this kind is called "the heat front". A strong isothermal discontinuity spreads behind this front, while the left-hand boundary expands into vacuum with an infinitely great speed. In the interval between the vacuum and the isothermal discontinuity all of the hydrodynamic parameters, and the temperature change monotonously. Examples of such solutions are shown on Fig.2 to 4 for the following values of parameters: $\gamma = 1.4$, $n = 0.2$, $\lambda = 3$, $A = 0.714 \times 10^6$ and $A = 0.714 \times 10^5$. The new variables appearing on Fig.2 to 6 are defined by

$$\alpha \int_{-\infty}^{1} \delta \, d\lambda = \int_{-\infty}^{\lambda} \delta \, d\lambda, \qquad z(1-\zeta) = \zeta$$

When $0 < A < A_*$, the temperature $\theta(\lambda)$ is no longer monotonous in the interval $(-\infty, \lambda_1)$, and temperature $\min \theta$ is reached at a certain point λ_2, where $-\infty < \lambda_2 < \lambda_1$, and $\theta'(\lambda_2) = 0$, which means that also the flow $q(\lambda_2) = 0$.

The hydrodynamic parameters ζ and δ reach their maximum values at a certain point λ_3. An analysis of Equations (1.4) shows that all of the parameters ζ, δ, θ, q change smoothly in the interval $(-\infty, \lambda_1]$, and that λ_3 is contained in the interval $(-\infty, \lambda_2]$.

An example of the second class of solutions is shown on Fig.5 and 6 for the following values of parameters: $\gamma = 1.4$, $n = 1$, $\lambda = 1$, $A = 0.625$. We may note that it had not been possible to derive this solution by the method set forth in Section 10. The basic system of Equations (1.1) was solved here by the method of finite differences, as described in [9].

10. Numerical integration of system (1.4) from point $\lambda = 1$ to point $\lambda = \lambda_1$ is carried out by the conventional point to point method. Computation in the interval $[\lambda^0, 1]$ in the indicated direction does not result in any loss of accuracy. Difficulties related to the loss of correct signs appear during numerical integration in the interval $(-\infty, \lambda_1]$. The boundary value problem arising here cannot be solved by the usual approximation method. With $A_* \leq A < \infty$ the following iteration process is used: with the given function $\theta(\lambda)$, using expansion (6.4), we integrate the equation of ζ' of (1.4) from $\lambda = -\infty$ to $\lambda = \lambda_1$, and then integrate equations of θ', δ' and q' with respect to the derived function $\zeta(\lambda)$ from $\lambda = \lambda_1$, to $\lambda = -\infty$, using the quadratic form (4.1) for solving the equation for δ'. This process is repeated until convergence is reached, when conditions at the left-hand side boundary (6.1) will be satisfied, while at the right-hand side boundary a certain value ζ_2^0, generally not equal to the known ζ_2, will be obtained with $\lambda = \lambda_1$. Parameter λ_1 is selected so as to have $\zeta_2^0 = \zeta_2$.

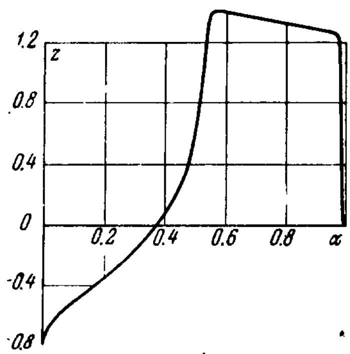

Fig. 6

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